

ON SINGULAR CASES IN THE DESIGN DERIVATIVE
OF GREEN'S FUNCTIONAL*

Robert Reiss
Howard University
Washington, D.C.

SUMMARY

This paper extends the author's prior development of a general abstract representation for the design sensitivities of Green's functional for linear structural systems to the case where the structural stiffness vanishes at an internal location. This situation often occurs in the optimal design of structures. Most optimality criteria require that optimally designed beams be statically determinate. For clamped-pinned beams, for example, this is possible only if the flexural stiffness vanishes at some intermediate location. The Green's function for such structures depends upon the stiffness and the location where it vanishes. A precise representation for Green's function's sensitivity to the location of vanishing stiffness is presented for beams and axisymmetric plates.

INTRODUCTION

This paper is concerned exclusively with the linear self-adjoint differential equation, represented in abstract form by

$$Lu = T^* E T u = f \quad \text{in } \Omega \quad (1a)$$

Here T and T^* are operators which are $L_2(\Omega)$ adjoints of each other, E is a stiffness operator which is symmetric with respect to the $L_2(\Omega)$ inner product, u is the response function and f is a specified disturbance. The open region $\Omega \in \mathbb{R}^n$ is bounded by $\partial\Omega$.

Appropriate mixed inhomogeneous boundary conditions are appended to equation (1a). These are

$$\begin{aligned} B Y u &= g \quad \text{on } \partial\Omega_1 \\ B^* Y^* E T u &= h \quad \text{on } \partial\Omega_2 \end{aligned} \quad (1b)$$

where $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$ and $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$. The operators Y and Y^* map functions in the domain of L into functions defined on $\partial\Omega_1$ and $\partial\Omega_2$, respectively. And the operators B and B^* map functions defined on $\partial\Omega_1$ and $\partial\Omega_2$, respectively, into functions defined on $\partial\Omega_1$ and $\partial\Omega_2$. Examples of the operators appearing in equations (1a) and (1b) can be found in reference 1 for a number of specific applications.

*This research was supported by the Howard University Large Space Structure Institute through Grant No. NAC-383 from the NASA Langley Research Center.

The stiffness operator E frequently depends upon one or more design parameters, which are collectively denoted by S . The operators T and T^* are generally differential operators which are independent of the design. The boundary operators may or may not be design dependent. There are two important classes of problems for which the boundary operators depend upon S . One class of such problems is usually referred to as shape optimization problems. Here, the boundary is the design variable, and consequently the boundary operators are necessarily design dependent. The other class of such problems occurs in structural optimization theory whenever optimality requires that the stiffness vanish somewhere in the interior of Ω . In this case, equation (1b) must also include an internal boundary where certain jump and/or continuity conditions are specified. This latter class of problems is the primary concern of this paper.

GREEN'S FUNCTION AND FUNCTIONAL

Oden and Reddy (ref. 2) have shown the operator P , which consists of the spatial operator of equation (1a) and the boundary operators of equation (1b), will be self-adjoint if the following integration by parts formula is satisfied

$$(Tu, ETv)_{\Omega} - (u, T^*ETv)_{\Omega} = (\gamma u, B^*\gamma^*ETv)_{\partial\Omega_2} - (B\gamma u, \gamma^*ETv)_{\partial\Omega_1} \quad (2)$$

for every u and v in the domain of P . In equation (2), (\cdot, \cdot) denotes the usual L_2 inner product and the appended subscript the domain of integration. Thus, for example, $(\cdot, \cdot)_{\partial\Omega}$ denotes the $L_2(\partial\Omega)$ inner product. In the remainder of this paper, it will be assumed that the operators specified in equations (1a) and (1b) do indeed satisfy equation (2).

The solution to equations (1a) and (1b) can now be obtained in terms of Green's function G , corresponding to the operator P , i.e.

$$u = (f, G)_{\Omega} + (g, \gamma^*ETG)_{\partial\Omega_1} + (h, \gamma G)_{\partial\Omega_2} \quad (3)$$

Equation (3) may be routinely derived by noting that $G(x, y)$ satisfies

$$T^*ETG(\cdot, y) = \delta_y \quad \text{in } \Omega \quad (4a)$$

and boundary conditions

$$\begin{aligned} B\gamma G(\cdot, y) &= 0 \quad \text{on } \partial\Omega_1 \\ B^*\gamma^*ETG(\cdot, y) &= 0 \quad \text{on } \partial\Omega_2 \end{aligned} \quad (4b)$$

where δ_y represents the Dirac distribution with a singularity at the location y . Upon taking the $L_2(\Omega)$ inner product of both sides of equation (4a) with u and integrating the result twice by parts according to equation (2), equation (3) immediately follows. Several illustrations of equation (3) have been derived by Roach (ref. 3) for specific operator equations.

Green's function $G(x, y)$ is defined on the Cartesian product space $\Omega \times \Omega$ and is generally singular when $x=y$. If any of the operators appearing in equation (4) depend upon the design variable(s) S , then G is a functional of the design S . Reiss (ref. 4) recently presented a compact formula for the

design derivative of G when E is the only operator appearing in equations (1a) and (1b) that depends upon the design S . For, in this case, let S and $S + \Delta S$ denote two designs and define ΔG by

$$\Delta G(x, y; S, \Delta S) = G(x, y; S + \Delta S) - G(x, y, S) \quad (5)$$

It immediately follows from equations (4a), (4b) and (5) that

$$\begin{aligned} T^* \Delta E \Delta G &= F & \text{in } \Omega \\ BY \Delta G &= 0 & \text{on } \partial \Omega_1 \\ B^* \gamma^* \Delta E \Delta G &= H & \text{on } \partial \Omega_2 \end{aligned} \quad (6)$$

where

$$\begin{aligned} F &= - T^* \Delta E T (G + \Delta G) \\ H &= - B^* \gamma^* \Delta E T (G + \Delta G) \end{aligned} \quad (7)$$

A cursory comparison of equation (6) with equation (1) shows that the solution for ΔG is immediately specified by equations (3) and (7); thus

$$\begin{aligned} \Delta G &= - (T^* \Delta E T (G + \Delta G), G)_{\Omega} \\ &\quad - (B^* \gamma^* \Delta E T (G + \Delta G), \gamma G)_{\partial \Omega_2} \end{aligned} \quad (8)$$

After applying the integration by parts formula (2), the variation ΔG simplifies to

$$\Delta G = - (T G, \Delta E T (G + \Delta G))_{\Omega} \quad (9)$$

Equation (9) is an integral equation for ΔG . Considerable simplification results if E is Gateaux differentiable with respect to the design. In this case, by restricting the design variation ΔS to be infinitesimal, ΔE is also infinitesimal and equation (9) may be linearized, i.e.,

$$\delta G = - (T G, \delta E T G)_{\Omega} \quad (10)$$

In equation (10), the symbol Δ has been replaced by δ in order to signify linearization. Equation (10) represents the design sensitivity of Green's functional.

SINGULAR DESIGNS

Beams

As stated at the outset, the primary focus of this paper is on designs whose stiffness vanishes somewhere in the interior of Ω . For beams whose boundary conditions are specified by (1b), the stiffness vanishes, at most, at two internal locations. Let x_0 denote the typical location for which $S(x_0) = 0$.

In terms of conventional notation for beams, the internal boundary condition at x_0 is the prescription of zero moment, while the matching condition is zero jump in both the shear force and the response. Thus

$$S(x) G_{xx}(x, y; x_0) \big|_{x=x_0} = 0 \quad (11a)$$

$$[[S(x)G_{xx}(x, y; x_0)]_x]_{x=x_0} = 0 \quad (11b)$$

$$[[G(x, y; x_0)]]_{x=x_0} = 0 \quad (11c)$$

where the subscripts denote partial differentiation with respect to the indicated arguments, and $[[\cdot]]$ denotes the jump in the quantity within the double brackets. In addition, at the extremities of the beam, G must meet the static and kinematic boundary conditions specified by equation (4b).

If the neighboring design $S + \delta S$ also vanishes at x_0 , then δS is specified by equation (10). If, however, $S + \delta S$ vanishes at $x_0 + \delta x_0$, then δG depends explicitly upon δx_0 as well as δS . Since the sensitivity of G with respect to δS is determined by equation (10), it remains to investigate the sensitivity of G to variations in x_0 .

With x_0 treated as the design variable, the counterpart to equation (5) becomes

$$\Delta G(x, y; x_0, \Delta x_0) \equiv G(x, y, x_0 + \Delta x_0) - G(x, y; x_0)$$

which, upon linearization, simplifies to

$$\delta G = G_{x_0}(x, y; x_0) \delta x_0 \quad (12)$$

It is important to note that G will generally have a slope discontinuity at x_0 , but $G + \delta G$ will have a slope discontinuity at $x_0 + \delta x_0$. It follows from equation (6) that δG satisfies

$$(S\delta G_{xx})_{xx} = 0 \quad 0 < x < x_0, \quad x_0 < x < L \quad (13)$$

plus appropriate homogeneous boundary conditions at $x=0$ and L . Due to the shift in the internal boundary x_0 , care must be taken in determining the internal matching conditions for δG . While G satisfies equations (11a), (11b) and (11c), $G + \delta G$ must satisfy

$$S(G + \delta G)_{xx} \big|_{x=x_0 + \delta x_0} = 0 \quad (14a)$$

$$[[S(G + \delta G)_{xx}]_x]_{x=x_0 + \delta x_0} = 0 \quad (14b)$$

$$[[G + \delta G]]_{x=x_0 + \delta x_0} = 0 \quad (14c)$$

Next, SG_{xx} is expanded in a Taylor series about x_0 to get

$$SG_{xx} \big|_{x_0 + \delta x_0} = SG_{xx} \big|_{x=x_0} + (SG_{xx})_x \big|_{x_0} \delta x_0$$

which, by virtue of equations (11a) and (14a), becomes

$$-S\delta G_{xx} \big|_{x=x_0} = (SG_{xx})_x \big|_{x=x_0} \delta x_0 \quad (15a)$$

Similarly, by expanding $(SG_{xx})_x$ and G about $x = x_0$, and making use of equations (11b), (11c), (13), (14b) and (14c), the following jump conditions are obtained:

$$[[S\delta G_{xx}]_x]_{x=x_0} = 0 \quad (15b)$$

$$[[\delta G]]_{x=x_0} = - [[G_x]]_{x=x_0} \delta x_0 \quad (15c)$$

The sensitivity δG is completely specified by equations (13), (15a), (15b), (15c) and the boundary conditions at $x=0$ and $x=L$. After multiplying both sides of equation (13) by G and integrating the result over the domains $0 < x < x_0$ and $x_0 < x < L$, it is found that

$$\begin{aligned} \delta G(z, y) = & [[G(x, z)(S(x)\delta G_{xx}(x, y))_x]]_{x=x_0} \\ & - [[G_x(x, z)S(x)\delta G_{xx}(x, y)]]_{x=x_0} \\ & + [[S(x)G_{xx}(x, z)\delta G_x(x, y)]]_{x=x_0} \\ & - [[(S(x)G_{xx}(x, z))_x\delta G(x, y)]]_{x=x_0} \end{aligned} \quad (16)$$

Equation (16) can be considerably simplified by making use of the jump conditions (11a,b,c) and (15a,b,c). The first term on the right hand side of equation (16) vanishes by virtue of equations (11c) and (15b); the third term also vanishes as a consequence of equation (11a). Now, substitution of equation (15a) into the second term, and equations (11b) and (15c) into the fourth term yield

$$\begin{aligned} \delta G(y, z) = & - \{ [[G_x(x, z)]]_{x=x_0} \bar{Q}(x_0, y) \\ & + [[G_x(x, y)]]_{x=x_0} \bar{Q}(x_0, z) \} \delta x_0 \end{aligned} \quad (17)$$

where $\bar{Q}(x_0, y)$ is the shear force at x_0 due to a unit load at y . Thus

$$\bar{Q}(x_0, y) = - (S(x) G_{xx}(x, y))_x|_{x=x_0} \quad (18)$$

The design derivative of Green's function, obtained from equations (12) and (18), becomes

$$\begin{aligned} \partial G(x, y; x_0) / \partial x_0 = & - [[G_x(x, z)]]_{x=x_0} \bar{Q}(x_0, y) \\ & - [[G_x(x, y)]]_{x=x_0} \bar{Q}(x_0, z) \end{aligned} \quad (19)$$

Axisymmetric Circular Plates

Thin isotropic elastic plates, like the elastic beams considered above, obey the fundamental equations (4a) and (4b). Consequently, the sensitivity of Green's function with respect to changes in the plate stiffness (thickness) must satisfy equation (10). For simplicity, only circular plates subject to axisymmetric loads and boundary conditions are considered. The plates may be full or annular, and the inner and outer boundaries of the plate will be denoted by a and b , respectively. For full plates, $a=0$. If the stiffness of the plate vanishes over a circle of radius r_0 and this radius is also a design parameter, then the sensitivity of G with respect to r_0 also must be determined.

At a circle of vanishing stiffness, the radial bending moment vanishes, and both the shear force and the response are continuous. Thus the counterparts to equations (11a), (11b) and (11c) are, respectively,

$$\begin{aligned} S(r)\{rG_{rr}(r,\zeta;r_0) + \nu G_r(r,\zeta;r_0)\}|_{r=r_0} &= 0 \\ [[r(S(r)(rG_{rr}(r,\zeta;r_0) + \nu G_r(r,\zeta;r_0)))_r \\ - S(r)(G_r(r,\zeta;r_0) + \nu rG_{rr}(r,\zeta;r_0))]]_{r=r_0} &= 0 \\ [[G(r,\zeta;r_0)]]_{r=r_0} &= 0 \end{aligned} \quad (20)$$

where ν is Poisson's ratio. And, of course, Green's function must still satisfy the mixed boundary conditions (4b).

Since r_0 is now the design variable, equation (12) is replaced by

$$\delta G = G_{r_0}(r,\zeta;r_0)\delta r_0 \quad (21)$$

where δr_0 denotes the infinitesimal shift in the location of vanishing stiffness.

It is desired to obtain an explicit representation for δG , analogous to equation (17). Toward this end, it is noted that δG satisfies $L\delta G=0$ and therefore

$$(S(r\delta G_{rr} + \nu\delta G_r))_{rr} - (S(r^{-1}\delta G_r + \nu\delta G_{rr}))_r = 0 \quad (22)$$

for $a < r < r_0$ and $r_0 < r < b$. Also δG satisfies the same boundary conditions at a and b as does G .

Before considering the jump conditions for δG at $r=r_0$, some notational simplification can be obtained by noting that $G(r,\zeta;r_0)$ represents the response at the circle of radius r due to a unit load distributed along the circle of radius ζ . Accordingly, let $\bar{M}(r,\zeta;r_0)$ and $\bar{Q}(r,\zeta;r_0)$ denote, respectively, the radial bending moment and shear force per unit length along the radius r due to the same unit load acting along the radius ζ . Thus equations (20) simplify to

$$r\bar{M}(r,\zeta;r_0)|_{r=r_0} = 0 \quad (23a)$$

$$[[\bar{Q}(r,\zeta;r_0)]]_{r=r_0} = 0 \quad (23b)$$

$$[[G(r,\zeta;r_0)]]_{r=r_0} = 0 \quad (23c)$$

For the sake of completeness, it is noted that \bar{M} and \bar{Q} are related to G through

$$\begin{aligned} r\bar{M} &= -S(rG_{rr} + \nu G_r) \\ r\bar{Q} &= -r(S(rG_{rr} + \nu G_r)) \\ &\quad + S(G_r + \nu rG_{rr}) \end{aligned} \quad (24)$$

For the varied design whose stiffness vanishes at $r_0 + \delta r_0$, the jump conditions analogous to (23a), (23b) and (23c) are, respectively,

$$r\bar{M}(r, \zeta; r_0) + r\delta\bar{M}(r, \zeta; r_0)|_{r=r_0+\delta r_0} = 0 \quad (25a)$$

$$[[r\bar{Q}(r, \zeta; r_0) + r\delta\bar{Q}(r, \zeta; r_0)]]_{r=r_0+\delta r_0} = 0 \quad (25b)$$

$$[[G(r, \zeta; r_0) + \delta G(r, \zeta; r_0)]]_{r=r_0+\delta r_0} = 0 \quad (25c)$$

By expanding $r\bar{M}|_{r=r_0+\delta r_0}$, $r\bar{Q}|_{r=r_0+\delta r_0}$ and $G|_{r=r_0+\delta r_0}$ in Taylor series about $r=r_0$, and simplifying the result using equations (25a), (25b) and (25c), the jump conditions

$$r\delta\bar{M}|_{r=r_0} = - (r\bar{M})_r|_{r=r_0} \delta r_0 \quad (26a)$$

$$[[r\delta\bar{Q}]]_{r=r_0+\delta r_0} = 0 \quad (26b)$$

$$[[\delta G]]_{r=r_0} = - [[G_r]]_{r=r_0} \delta r_0 \quad (26c)$$

are readily obtained. The quantities $\delta\bar{M}$ and $\delta\bar{Q}$ are implicitly defined through equations (24). Thus

$$\begin{aligned} r\delta\bar{M} &= - S(r\delta G_{rr} + v\delta G_r) \\ r\delta\bar{Q} &= - r(S(r\delta G_{rr} + v\delta G_r))_r \\ &\quad + S(\delta G_r + vr\delta G_{rr}) \end{aligned} \quad (27)$$

The sensitivity δG may now be determined explicitly by multiplying both sides of equation (22) by G and integrating the result from $r=a$ to $r=b$. Thus

$$\begin{aligned} \delta G(\zeta, \xi) &= + [[r\bar{Q}(r, \zeta)\delta G(r, \xi)]]_{r=r_0} \\ &\quad - [[r\bar{M}(r, \zeta)\delta G_r(r, \xi)]]_{r=r_0} \\ &\quad + [[G_r(r, \zeta) r\delta\bar{M}(r, \xi)]]_{r=r_0} \\ &\quad - [[G(r, \zeta)r\delta\bar{Q}(r, \xi)]]_{r=r_0} \end{aligned} \quad (28)$$

The second and fourth terms on the right hand-side of equation (28) vanish by virtue of equation (23a) and equations (23c) and (26b), respectively. Moreover, equations (26c) and (26b) transform the first term into

$$- r\bar{Q}(r, \zeta)|_{r=r_0} [[G_r(r, \xi)]]_{r=r_0} \delta r_0$$

while the third term, obtained from equation (26a) and the equilibrium equation, becomes

$$- r\bar{Q}(r, \xi)|_{r=r_0} [[G_r(r, \zeta)]]_{r=r_0} \delta r_0$$

Therefore, the design derivative $\partial G/\partial r_0$ is given by

$$\begin{aligned} \partial G(\zeta, \xi; r_0)/\partial r_0 &= - \{ r_0\bar{Q}(r_0, \zeta)[[G_r(r, \xi)]]_{r=r_0} \\ &\quad + r_0\bar{Q}(r_0, \xi)[[G_r(r, \zeta)]]_{r=r_0} \} \end{aligned} \quad (29)$$

The usual method of obtaining structural optimality criteria associated with specific cost functionals relies on developing an appropriate variational formulation of the field equations (1a) and (1b). Moreover, each cost functional requires a different variational formulation. In contrast to the historical approaches, the design derivatives specified by equations (10) and (17) or (29) can be used directly to determine the optimality criteria associated with any cost functional without the need of a variational formulation. In order to illustrate the foregoing claim, structural optimality criteria will be derived for two different cost functionals: minimum response and minimum compliance.

The optimality criteria associated with the design of a fixed-weight structure for minimum response at a specified location is considered first. Boundary conditions and loads are assumed known. It is desired to obtain a complete description of the design variable S including its singular points (locations of vanishing stiffness). Shield and Prager (ref. 5) obtained the optimality condition for this problem only after discovering the principle of stationary mutual potential energy. They did not address the question of locating the singular points. However, at least one author (ref. 6) incorrectly assumed that such points can be obtained by requiring the response to be continuously differentiable everywhere.

Attention is now directed toward equation (1a) and (1b) with $g = h = 0$. According to equation (3), the solution for the response is

$$u = (f, G)_{\Omega}$$

Let the location y be specified and $u(y)$ be a minimum. Thus

$$u(y) = (f(\cdot), G(\cdot, y))_{\Omega} \quad (30)$$

whence

$$\delta u(y) = (f(\cdot), \delta G(\cdot, y))_{\Omega} \quad (31)$$

For the moment, it will be assumed that S is not singular anywhere. After substituting equation (10) into equation (31) and changing the order of the resulting double integration, equation (31) becomes

$$\delta u(y) = - (Tu, \frac{\partial E}{\partial S} \delta STG(\cdot, y))_{\partial \Omega} \quad (32)$$

The volume constraint may be easily handled through a Lagrange multiplier. Let $v(S)$ denote the specific volume and λ a Lagrange multiplier. Then the condition $\delta u(y) = 0$ for all designs consistent with the constant volume constraint requires that the augmented functional

$$- (Tu, \frac{\partial E}{\partial S} \delta STG(\cdot, y))_{\partial \Omega} + \lambda (1, \frac{\partial v}{\partial S} \delta S)_{\Omega} = 0$$

for all variations δS . Thus the optimality condition

$$Tu \cdot \frac{\partial E}{\partial S} \cdot TG(\cdot, y) = \lambda \frac{\partial v}{\partial S} \quad (33)$$

follows immediately. For beams, the equivalent to equation (33) was obtained in reference 5. The constant λ appearing in equation (33) can be determined from the fixed-volume constraint.

It was stated earlier that, in many applications, there can be no solution to the optimality and field equations unless a singular point occurs within the structure. In this case it is not possible to determine the location of the singularity from the optimality condition and field equations. This location must be considered an additional design variable, and consequently its location will be determined from an additional optimality condition.

For simplicity, it will be assumed that the structure is a beam. In this case, equation (17) is substituted into equation (31), and the resulting double integrals are evaluated by reversing the order of the integrations. Thus

$$\begin{aligned} \delta u(y) = & - Q(x_0)[[G_z(z,y)]]_{z=x_0} \delta x_0 \\ & - \bar{Q}(x_0,y)[[u_z(z)]]_{z=x_0} \delta x_0 \end{aligned} \quad (34)$$

Since the specific volume v is independent of x_0 , the optimality condition to determine x_0 is obtained directly from equation (34). Thus

$$\begin{aligned} Q(x_0)[[G_z(z,y)]]_{x=x_0} \\ + \bar{Q}(x_0,y)[[u_z(z)]]_{x=x_0} = 0 \end{aligned} \quad (35)$$

Next, consider the problem of minimizing the compliance of a structure. The compliance C is defined to be the work done by the external loads. Thus

$$C = (u, f)_{\Omega}$$

whence

$$\delta C = (\delta u, f)_{\Omega} \quad (36)$$

Substitution of equation (32) into equation (36) yields

$$\delta C = - (Tu, \frac{\partial E}{\partial S} \delta S Tu)_{\Omega}$$

Consequently, the optimality condition for prescribed volume becomes

$$Tu \cdot \frac{\partial E}{\partial S} \cdot Tu = \lambda \frac{\partial v}{\partial S} \quad (37)$$

Equation (37), in its various specific forms, has been derived by many authors for specific structures. In virtually all instances, the principle of minimum potential energy has been an ingredient necessary to the derivation.

The location of any singular points may be determined in the same way that it was done for the minimum response design. In this case, equation (34) is substituted into equation (36) to yield

$$\delta C = - 2Q(x_0)[[u_z(z)]]_{z=x_0} \delta x_0$$

The optimality condition to determine x_0 , therefore, is

$$[[u_z(z)]]_{z=x_0} = 0 \quad (38)$$

As a final remark, it is pointed out, without elaboration, that the approach taken in this paper is easily generalized to transient structures.

REFERENCES

1. Reiss, R.; and Haug, E.J.: Extremum Principles for Linear Initial-Value Problems of Mathematical Physics. Int. J. Engrg. Sci., vol. 16, 1978, pp. 231-251.
2. Oden, J.T.; and Reddy, J.N.: Variational Methods in Theoretical Mechanics. Springer-Verlag, Berlin, 1976.
3. Roach, G.F.: Green's Functions: Introductory Theory with Applications. Van-Nostrand Reinhold, London, 1970.
4. Reiss, R.: On the Design Derivative of Green's Functional. Eighteenth Mid-western Mechanics Conference, Iowa City, May 1983.
5. Shield, R.T.; and Prager, W.: Optimal Structural Design for Given Deflection J. App. Math. Phys. (ZAMP), vol. 21, 1970, pp. 513-523.
6. Huang, N.C.; and Tang, H.T.: Minimum-Weight Design of Elastic Sandwich Beams with Deflection Constraints, vol. 4, 1969, pp. 277-298.